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HW6

PHYS 515B
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1. A linearly polarized plane electromagnetic wave of frequency ω and intensity I_0 is scattered by a free electron. Derive the cross section for scattering in the non-relativistic limit. Find the angular distribution and polarization of the scattered radiation.

   Solution. The electric intensity of the incident plane wave at the electron is $\vec{E} = \vec{E}_0 e^{-i\omega t}$ and the equation of motion is

$$m\dot{\vec{v}} = -e\vec{E}.$$

   The rate of the radiation emitted by the electron at angle α with the direction of acceleration is

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{e^2\dot{\vec{v}}^2}{4\pi c^3}\sin^2\alpha \\ &= \frac{e^4\vec{E}^2}{8\pi m^2 c^3}\sin^2\alpha. \end{aligned}$$

   If the z-axis is along the direction of the incident waves, x-axis is in the plane containing the z-axis and $\vec{r}$, the direction of the scattered waves, we have that

$$\cos\alpha = \sin\theta\cos\phi.$$

   We have that the initial intensity is $\frac{cE_0^2}{8\pi}$ Combine this fact with the angular relation and the previous equation for $\frac{dP}{d\Omega}$ to get

$$\begin{aligned} \frac{dP}{d\Omega} &= I_0\left(\frac{e^2}{mc^2}\right)^2(1-\sin^2\theta\cos^2\phi) \\ \therefore \frac{d\sigma}{d\Omega} = \frac{dP}{I_0 d\Omega} &= \left(\frac{e^2}{mc^2}\right)^2(1-\sin^2\theta\cos^2\phi). \end{aligned}$$

2. Calculate the cross section of Thomson scattering for high frequency electromagnetic waves. In this case, the radiation damping should be taken into account.

   Solution. The equation of motion is

$$m\ddot{\vec{x}} = q\vec{E}_0 e^{-i\omega t} + \frac{2q^2}{3c^3}\dddot{\vec{x}}.$$

   Consider small damping, that is $\dddot{\vec{x}} = -\omega_0^2\dot{\vec{x}}$. Now combine this into the equation of motion,

$$\begin{aligned} \ddot{\vec{x}} + \gamma\dot{\vec{x}} &= \frac{q}{m}\vec{E}_0 e^{-i\omega t} \\ \text{where } \gamma &= \frac{2q^2\omega_0^2}{3mc^3}. \end{aligned}$$

Let $\vec{x} = \vec{x}_0 e^{-i\omega t}$. Now we have that

$$\vec{x}_0 = -\frac{q\vec{E}_0}{m\omega_0(\omega + i\gamma)}.$$

The radiation field of an electron at a point $\vec{r}$ from the electron is

$$\vec{E} = \frac{q}{e^2 r}\hat{r} \times (\hat{r} \times \ddot{\vec{r}}).$$

Let α be the same as in the previous problem, we have that

$$\vec{E} = -\frac{q^2\omega\vec{E}_0 \sin\alpha}{mc^2(\omega + i\gamma)r}e^{-i\omega t}.$$

The intensity of the waves is

$$\begin{aligned} I = \frac{c}{4\pi}\langle|\vec{E} \times \vec{B}|\rangle &= \frac{c}{8\pi}\Re(E^* E) \\ &= \frac{\omega^2}{\omega^2 + \gamma^2}\frac{r_0^2}{r^2}I_0 \sin^2\alpha \\ &= \frac{\omega^2}{\omega^2 + \gamma^2}\frac{r_0^2}{r^2}I_0(1 - \sin^2\theta\cos^2\phi). \end{aligned}$$

where $r_0 = \frac{e^2}{mc^2}$ and $I_0 = \frac{cE_0^2}{8\pi}$, intensity of the incident waves.
If the incident waves are not polarized, ϕ is random, we have

$$\begin{aligned} \langle I(\theta)\rangle &= \frac{\omega^2}{\omega^2 + \gamma^2}\frac{r_0^2}{r^2}I_0\frac{1}{2\pi}\int_0^{2\pi}(1 - \sin^2\theta\cos^2\phi)d\phi \\ &= \frac{1}{2}\frac{\omega^2}{\omega^2 + \gamma^2}\frac{r_0^2}{r^2}I_0(1 + \cos^2\theta). \end{aligned}$$

The total scattered power is

$$\begin{aligned} P &= \int_0^{\pi}\langle I(\theta)\rangle 2\pi r^2 \sin\theta d\theta \\ &= \frac{8\pi}{3}\frac{\omega^2}{\omega^2 + \gamma^2}r_0^2 I_0. \end{aligned}$$

The scattering cross section is

$$\sigma = \frac{P}{I_0} = \frac{8\pi}{3}\frac{\omega^2}{\omega^2 + \gamma^2}r_0^2.$$

3. Calculate the scattering cross section of a plane electromagnetic wave by a charged harmonic oscillator, i.e. when there is no incident waves, the equation of motion of the charged particle is $\ddot{\mathbf{r}} + \omega_0^2\mathbf{r} = 0$, where $\mathbf{r}$ is the coordinate vector of the particle, and ω_0 is the frequency of the harmonic motion.

Solution. Take for the equation of the electric field caused by a plane EM wave

$$\vec{E} = \vec{E}_0 \cos(\omega t + \delta)$$

for some frequency ω and phase δ.

The equation of motion for the particle with the introduction of the plane wave is

$$\ddot{\vec{r}} + \omega_0^2 \vec{r} = \frac{e}{m} \vec{E}_0 \cos(\omega t + \delta).$$

Solve for $\vec{r}$

$$\vec{r} = \frac{e\vec{E}_0 \cos(\omega t + \delta)}{m(\omega_0^2 - \omega^2)}$$

Calculate the dipole moment $\vec{d}$ and combine into equation for scattering cross section using formulas from Landau and Lifshitz

$$d\sigma = \left(\frac{e^2}{mc^2}\right)^2 \frac{\omega^4}{(\omega^2 - \omega_0^2)^2} \sin^2 \theta d\Omega$$

where θ is the angle the plane wave is scattered through.